SEM-4 (MATHEMATICS GENERAL)

ANUP RAY

Theorem 5 Prove that $G = \bigcup_{a\in G} Ha$ or $G = \bigcup_{a\in G} aH$

That is, G is the union of all right cosets of H in G.
(G is the union of all left cosets of H in G)

Note: In case of additive group $G = \bigcup_{a\in G} (H+a)$
or $G = \bigcup_{a\in G} (a+H)$

Proof: $Ha \subseteq G \ \forall a \in G$

$\Rightarrow \bigcup_{a\in G} Ha \subseteq G$ ---- (1)

Again $a \in G \Rightarrow a \in Ha$

That is every element of G is contained in a right coset of H in G.

Hence $G \subseteq \bigcup_{a\in G} Ha$ ---- (2)

From (1) & (2) we have $G = \bigcup_{a\in G} Ha$

Similarly we can prove $G = \bigcup_{a\in G} aH$.

Theorem 6 Right coset decomposition of a group $(G,\cdot)$
If a relation ρ be defined by $a\rho b \Leftrightarrow ab^{-1} \in H$, $\forall a, b \in G$, then
(i) ρ is an equivalance relation.
(ii) $Ha = [a]$, equivalance class determined by $a \in G$ and hence $G = \bigcup_{a\in G} Ha$.

Proof: $a\rho b \Leftrightarrow ab^{-1} \in H$

Now $aa^{-1} = e \in H \Rightarrow a\rho a \Rightarrow \rho$ is reflexive

Let $a\rho b$, then $ab^{-1} \in H$

$ab^{-1} \in H \Rightarrow (ab^{-1})^{-1} \in H \Rightarrow (b^{-1})^{-1} a^{-1} \in H$

$\Rightarrow ba^{-1} \in H \Rightarrow b\rho a \Rightarrow \rho$ is symmetric.

Let $a\rho b$ and $b\rho c$. Then $ab^{-1} \in H$, $bc^{-1} \in H$.

By closure property of H, $(ab^{-1})(bc^{-1}) \in H$
$\Rightarrow a(b^{-1}b)c^{-1} \in H \Rightarrow (ae)c^{-1} \in H$
$\Rightarrow ac^{-1} \in H \Rightarrow a\rho c \Rightarrow \rho$ is transitive.

Since ρ is symmetric, reflexive and transitive, it follows that ρ is an equivalance relation.

The above equivalance relation ρ partitions G into mutually disjoint equivalance classes $[a]$, $a \in G$

Then we have to show that $[a] = Ha$.
Let $x \in [a]$. Then $x \in [a] \Rightarrow x\rho a$
$\Rightarrow xa^{-1} \in H \Rightarrow x \in Ha \Rightarrow [a] \subseteq Ha$ ---(1)

Again if $y \in Ha$, then $y \in Ha \Rightarrow y = ha$ for some $h \in a \Rightarrow ya^{-1} = h \in H \Rightarrow y\rho a$
$\Rightarrow y \in [a] \Rightarrow Ha \subseteq [a]$ -- (2)

Thus from (1) & (2) we get $[a] = Ha$
$\Rightarrow G$ is the union of mutually disjoint right cosets of H in G.

All the right cosets of H in G are said to form the right coset decomposition of G.

Note: Similarly we can prove for left cosets of H in G if ρ is defined by $a\rho b \Rightarrow a^{-1}b \in H$, $\forall\, a, b \in G$.